DEFINABILITY IN PHYSICS

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Abstract. The concept of definability of physical fields in a set-theoretical foundation is introduced. A set theory is selected in which we get mathematics enough to produce a nonlinear sigma model. Quantization of the model requires only a null postulate and is then shown to be necessary and sufficient for definability in the theory. We obtain scale invariance and compactification of the spatial dimensions effectively. Three interesting examples of the relevance to physics are suggested.

We look to provide a deep connection between physics and mathematics by requiring that physical fields must be definable in a set-theoretical foundation. The well-known foundation of mathematics is the set theory called Zermelo-Fraenkel (ZF). In ZF, a set U of finite integers is definable if and only if there exists a formula $\Phi_U(n)$ from which we can unequivocally determine whether a given finite integer n is a member of U or not. That is, when a set of finite integers is not definable, then there will be at least one finite integer for which it is impossible to determine whether it is in the set or not. Other sets are definable in a theory if and only if they can be mirrored by a definable set of finite integers. Most sets of finite integers in ZF are not definable. Furthermore, the set of definable sets of finite integers is itself not definable in ZF. [1]

A physical field in a finite region of space is definable in a set-theoretical foundation if and only if the set of distributions of the field's energy among eigenstates can be mirrored in the theory by a definable set of finite integers. This concept of definability is appropriate because, were there such a physical field whose set of energy distributions among eigenstates was mirrored by an undefinable set of finite integers, that field would have at least one energy distribution whose presence or absence is impossible to determine, so the field could not be observable. Therefore, our task is to find a foundation in which it is possible to specify completely the definable sets of finite integers and to construct the fields mirrored by these sets.

The definable sets of finite integers cannot be specified completely in ZF because there are infinitely many infinite sets whose definability is undecidable. So we will start with a sub-theory containing no infinite sets of finite integers. Then all sets of finite integers are *ipso facto* definable. This will mean, of course, that the set of all finite integers, called ω, cannot exist in that sub-theory. The set ω exists in ZF directly in consequence of two axioms: an axiom of infinity and an axiom schema of subsets. Thus, we must delete one or the other of these axioms. If we delete the axiom of infinity we will then have no need for the axiom schema of subsets either since all sets are finite. However that theory is too poor to obtain the functions of a real variable necessary for physical fields. So the task reduces to whether or not, starting by deleting the axiom schema of subsets from ZF while retaining the axiom of infinity, we can build a foundation in which we obtain just those physical fields mirrored by the definable sets of finite integers.

In the appendix we show eight axioms. The first seven are the axioms of ZF except that the axiom schema of replacement has been modified. The usual replacement axiom (AR) asserts that, for any functional relation, if the domain is a set, then the range is a set. That axiom actually combines two independent axioms: the axiom schema of subsets, which we wish to delete, and an axiom schema of bijective replacement (ABR), which refers only to a one-to-one functional relation. Accordingly, we can delete the axiom schema of subsets from ZF by substituting ABR for AR, forming the sub-theory ZF–AR+ABR.

We shall first discuss how ZF–AR+ABR differs from ZF. To do this, we look at the axiom of infinity. The axiom of infinity asserts the existence of at least one set ω* that contains, in general, infinite as well as finite ordinals. There are actually infinitely many such sets. In ZF, we obtain the minimal ω*, a set with just all the finite ordinals called ω, by using the axiom schema of subsets to provide the intersection of all the sets created by the axiom of infinity. However, in ZF–AR+ABR, without the axiom schema of subsets, this minimal set ω cannot be obtained and therefore every provable statement must hold for any ω^*. A member of ω^* is an "integer". An "infinite integer" is a member that maps one-to-one with ω^*. A "finite integer" is a member that is not an infinite integer. Also, in ZF–AR+ABR, a set of finite integers is finite. We denote finite integers by *i, j, k,* ℓ*, m* or *n*.

We now adjoin to ZF–AR+ABR an axiom asserting that all sets of integers are constructible. By constructible sets we mean sets that are generated sequentially by some process, one after the other, so that the process well-orders the sets. Goedel has shown that an axiom asserting that all sets are constructible can be added to ZF, giving a theory usually called ZFC^+.[2] It has also been shown that no more than countably many constructible sets of integers can be proven to exist in ZFC^+.[3] This result holds for the sub-theory ZFC^+–AR+ABR. Therefore we can adjoin to ZF–AR+ABR a new axiom asserting that all the subsets of ω^* are constructible and that its power set is countable. We refer to these eight axioms as theory T.

As required for consistency, Cantor's proof or its equivalent cannot be carried out in T [4] so only countable sets can be obtained. Since all sets are countable, the continuum hypothesis holds. However, as the axiom schema of subsets is not available, we cannot prove the induction theorem, so not all countable sets that exist in ZF can exist in T. For example, we cannot have infinite series, whereas in ZF infinite series play an important role in the development of mathematics. Nevertheless, using our axiom of constructibility we can obtain some functions of a real variable.

We first show that the theory T contains a real line, obtainable by a non-standard approach. Recall the definition of "rational numbers" as the set of ratios of any two members of the set ω, usually called **Q**. In T, we can likewise establish for ω^* the set of ratios of any two of its integers, finite or infinite, as an "enlargement" of the rational numbers and we shall call this enlargement **Q***.

Two members of **Q*** are called "identical" if their ratio is 1. We now employ the symbol "≡" for "is identical to". An "infinitesimal" is a member of **Q*** "equal" to 0, i.e., letting y signify the member and employing the symbol "=" to signify equality, $y = 0 \leftrightarrow \forall k[y < 1/k]$. The reciprocal of an infinitesimal is "infinite". A member of **Q*** that is not an infinitesimal and not infinite is "finite", i.e., $[y \neq 0 \wedge 1/y \neq 0] \leftrightarrow \exists k[1/k < y < k]$. We apply this concept of equality to the interval between two finite members of **Q***; two finite members are either equal or the interval between them is finite. The constructibility axiom in T well-orders the power set of ω*, creating a metric space composed of all the subsets of ω*. These subsets represent the binimals making up a real line **R***. [5]

Equality-preserving bijective mappings between finite intervals of **R*** are homeomorphic, i.e., bijective mappings $\phi(x,u)$ of an interval X onto an interval U in which $x \in X$ and $u \in U$ such that $\forall x_1,x_2,u_1,u_2[\phi(x_1,u_1) \wedge \phi(x_2,u_2) \rightarrow [x_1 = x_2 \leftrightarrow u_1 = u_2]]$ will produce function pieces that are biunique and continuous. These "biunique pieces" will now be joined sequentially to obtain functions of a real variable suitable for construction of physical fields.

We define a "function of a real variable in T" as either a constant (which is obtained directly from ABR.) or a continuously connected sequence of biunique pieces such that its derivative is also a function of a real variable in T. These hereditarily defined functions obviously are extremely smooth and of bounded variation. They are, moreover, restricted only to polynomials, since infinite series do not exist in T. We have thus been able to develop a theory of polynomials. We shall focus now on generating polynomials by an algorithm that is uniformly convergent to the solutions of the physically relevant Sturm-Liouville problem, written here in its integral form:

$$\int_a^b [p\left(\frac{du}{dx}\right)^2 - qu^2]dx = \lambda\int_a^b ru^2dx \quad \text{where } a \neq b,\ u\frac{du}{dx} \equiv 0 \text{ at a and b} \qquad (1)$$

and p, q and r are functions of x.

Characteristic solutions to this integral equation are obtained by minimizing λ for $\int_a^b ru^2dx$ constant. This allows an algorithm generating increasingly higher degree polynomials u_n, where n denotes the n[th] iteration, such that ∀k∃n [$\int_a^b [p\left(\frac{du_n}{dx}\right)^2 - qu_n{}^2]dx - \lambda_n \int_a^b ru_n{}^2 dx$ < 1/k]. We refer to polynomials of sufficiently high degree as an "eigenfunction". Every eigenfunction, since it is a polynomial, can be decomposed into "irreducible biunique eigenfunction pieces", as required in T.

We now show this theory is a foundation for fields governed by a nonlinear sigma model. Let us first consider two eigenfunctions, $u_1(x_1)$ and $u_2(x_2)$; for each let $p \equiv 1$, $q \equiv 0$ and $r \equiv 1$ and we shall call x_1 "space" and x_2 "time". It is well known that $\left(\frac{\partial u_1 u_2}{\partial x_1}\right)^2 - a\left(\frac{\partial u_1 u_2}{\partial x_2}\right)^2$ is the Lagrange density for a one-dimensional string and, by minimizing the integral of this function over all space and time, i.e., by Hamilton's principle, we can determine the field equations. We immediately extend this to separable bosonic strings in finitely many space-like (i) and time-like (j) dimensions. Extension to matrix fields is also possible. Since they are functions of real variables in T, the fields will be continuous, differentiable to all orders and of bounded variation, thus having no singularities.

Let $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ be eigenfunctions with non-negative eigenvalues $\lambda_{\ell mi}$ and $\lambda_{\ell mj}$, respectively. We define a "field" as a sum of eigenstates $\Psi_m = \Psi_{1m} + i\Psi_{2m}$, $\Psi_{\ell m} = \prod_i u_{\ell mi} \prod_j u_{\ell mj}$, subject to the postulate that for every eigenstate m the value of the integral of the Lagrange density over $d\tau$, where $d\tau = \prod_i r_i dx_i \prod_j r_j dx_j$, is *identically* null:

$$\sum_{\ell} \int \left\{ \sum_i \frac{1}{r_i}\left[P_{\ell mi}\left(\frac{\partial \Psi_{\ell m}}{\partial x_i}\right)^2 - Q_{\ell mi}\Psi_{\ell m}^2\right] - \sum_j \frac{1}{r_j}\left[P_{\ell mj}\left(\frac{\partial \Psi_{\ell m}}{\partial x_j}\right)^2 - Q_{\ell mj}\Psi_{\ell m}^2\right]\right\} d\tau \equiv 0 \;\; \textit{for all m.} \quad (2)$$

In this integral equation the P and Q can be functions of any of the x_i and x_j, thus of any $\Psi_{\ell m}$ as well. This is a *nonlinear sigma model.* The $\Psi_{\ell m}$ are given by a generalized Sturm-Liouville algorithm [6].

For equation (2), we can *prove quantization*. Since they are identical, we will represent both

$$\sum_m \sum_\ell \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] \right\} d\tau \text{ and } \sum_m \sum_\ell \int \left\{ \sum_j \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \right\} d\tau \text{ by } \alpha:$$

I. α is positive and must be closed to addition and to the absolute value of subtraction; thus in T α is an integer times a constant that is either infinitesimal or finite.

II. There is either no field (in which case $\alpha \equiv 0$) or otherwise in T α is non-infinitesimal, (in which case $\alpha \neq 0$); thus $\alpha = 0 \leftrightarrow \alpha \equiv 0$.

III. $\therefore \alpha \equiv nI$, where n is an integer and I is a finite constant such that $\alpha = 0 \leftrightarrow n \equiv 0$.

Equation (2) is in the form of a generalized Klein-Gordon equation. If we have finitely many space-like dimensions and only one time dimension, it is also a generalized Schroedinger equation. Let $\Psi = u_{\mathrm{r}}(\mathbf{r})[u_1(\mathrm{t}) + \mathrm{i}u_2(\mathrm{t})]$ with the normalization $\int \Psi\Psi^* \mathrm{d}\mathbf{r} \equiv 1$. Differentiating $\int \Psi\Psi^* \mathrm{d}\mathbf{r}$ with respect to time results in $\frac{du_1}{dt} = -\,\omega u_2$ and $\frac{du_2}{dt} = \omega u_1$ or $\frac{du_1}{dt} = \omega u_2$ and $\frac{du_2}{dt} = -\,\omega u_1$. We can now show $\hbar/2\mathrm{i}\left[\Psi^*\left(\frac{\partial \Psi}{\partial t}\right) - \left(\frac{\partial \Psi^*}{\partial t}\right)\Psi\right] = \hbar/\omega\left[\left(\frac{du_1}{dt}\right)^2 + \left(\frac{du_2}{dt}\right)^2\right] u_{\mathrm{r}}^2(\mathbf{r})$. The l.h.s. is the usual time term in the Lagrange density for a Schroedinger equation. The r.h.s. is how it translates for equation (2). Thus, identifying I as the action integral over an irreducible biunique time-eigenfunction piece and using $\int_{piece} \omega\, dt \equiv \frac{\pi}{2}$, we get $\hbar \frac{\pi}{2} \equiv I$. If n_m is the number of biunique pieces of the m^{th} eigenfunction in time period ϑ, it follows that the energy in the m^{th} eigenstate is in quanta $n_m I/\vartheta$ and the field energy, the sum of energies in all of the eigenstates, is MI/ϑ, where $M \equiv \sum j_m\, n_m$ and j_m is the number of quanta in the m^{th} eigenstate. MI is then an action α for the field. The following results are scale invariant.

We have now set the stage for our discussion of the definability of fields in the theory T. Recall that every set (in T) of finite integers is finite and hence definable. Thus a physical field can be definable in T if and only if the set of all distributions of field energy among eigenstates can be mirrored by a set (in T) of finite integers. We now show that the field of equation (2) with finitely many space-like dimensions and one time dimension in a finite region of space is definable in T.

For a given finite α, the ordered set of the j_m corresponding to each distribution of energy among the eigenstates of the field can always be mapped to a different finite integer, e. g.,

$$\left\{ j_m \,|\, \sum_m^M j_m n_m \equiv M \right\} \Rightarrow \prod_m^M \left(P_m\right)^{j_m} \quad \text{where } P_m \text{ is the } m^{th} \text{ prime starting with 2.} \tag{3}$$

In T, the set of these finite integers for all distributions of energy exists and, moreover, every set of finite integers will mirror a distribution for some α. Thus quantization is sufficient for definability. Furthermore, we can show that quantization is also necessary for definability in T. Given a finite α, if I were infinitesimal, then the integer M would have to be infinite. In that case, the set of all distributions of energy among eigenstates cannot be mirrored by any set (in T) of finite integers. Therefore, *quantization is necessary and sufficient for definability in T*. By analogous reasoning, definability in T is equivalent to compactification of the spatial dimensions [7].

In addition providing a foundation for definable fields, here are three interesting examples of the relevance of theory T to physics. First, as recently reported by Borchers and Sen [8], a countable space-time structure in which physical fields are locally homeomorphic to the real line, as in T, will satisfy the stringent topological requirements of Einstein causality. Second, the problem suggested by Dyson [9], that the power series expansions essential in quantum electrodynamics are actually divergent so must be only asymptotic expansions that in practice give an accurate approximation, is adequately handled in a theory where all power series are finite. Thus this foundation encompasses a possible convergence of gravitational and quantum theories. Third, the deep question raised by Wigner [10] about the unreasonable effectiveness of mathematics in physics is answered directly.

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1. Tarski, A., Mostowski, A., Robinson, R., *Undecidable Theories*. North Holland, Amsterdam, 1953.

2. Goedel, K., The consistency of the axiom of choice and of the generalized continuum hypothesis. *Annals of Math Studies*, 1940, 3.

3. Cohen, P. J., *Set Theory and the Continuum Hypothesis*, New York, 1966.

4. The axiom schema of subsets is $\exists u[[u = 0 \vee \exists x x \in u] \wedge \forall x x \in u \leftrightarrow x \in z \wedge \boldsymbol{X}(x)]$, where z is any set and $\boldsymbol{X}(x)$ is any formula in which x is free and u is not free. The axiom enters ZF in AR but can also enter in the strong form of the axiom of regularity. (Note T has the weak form.) This axiom is essential to obtain the diagonal set for Cantor's proof, using $x \notin f(x)$ for $\boldsymbol{X}(x)$, where f(x) is an assumed one-to-one mapping between ω* and P(ω*). The argument leads to the contradiction $\exists c \in z \boldsymbol{X}(c) \leftrightarrow \neg \boldsymbol{X}(c)$, where f(c) is the diagonal set. In ZF, this denies the mapping exists. In T, the same argument instead denies the existence of the diagonal set, whose existence has been hypothesized while the mapping was hsserted as an axiom. What if we tried another approach for Cantor's proof, by using ABR to get a characteristic function? Let $\phi(x,y) \leftrightarrow [\boldsymbol{X}(x) \leftrightarrow y = (x,1) \wedge \neg \boldsymbol{X}(x) \leftrightarrow y = (x,0)]$ and z = ω*. If c were a member of ω*, t = (c,1) and t = (c,0) both lead to a contradiction. But, since the existence of the diagonal set f(c) is denied and since a one-to-one mapping between ω* and P(ω*) is an axiom, as f(c) is not a member of P(ω*), so c cannot be a member of ω*. In T the characteristic function exists but has no member corresponding to a diagonal set.

5. The axiom of constructibility generates sequentially all the subsets of ω* in a set of ordered pairs. The left-hand member of each pair is a subset of ω* and the right-hand member is an integer indicating the order in which it was generated. If we let the integers not present in each subset be a "1" in the corresponding binimal and the integers that are present be a "0", then the right-hand member is the magnitude of that binimal and serves as a distance measure on the line R*.

6. The $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ are iterated using (1). The $p_{\ell mi}(x_i), q_{\ell mi}(x_i), p_{\ell mj}(x_j)$ and $q_{\ell mj}(x_j)$ will generally change at each iteration and are given by $p_{\ell mi} = \int \frac{P_{\ell mi}\Psi^2_{\ell m}\,d\tau}{u^2_{\ell mi} r_i\,dx_i} \Big/ \int \frac{\Psi^2_{\ell m}\,d\tau}{u^2_{\ell mi} r_i\,dx_i}$, etc.
Since the field is continuous, differentiable to all orders, of bounded variation and thus free of singularities, iterations for all $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ will converge jointly within a finite region.

7. The same reasoning can be applied to the spatial dimensions. The field of equation (2) is definable in T if and only if *M* is finite. In T, the range and domain of the irreducible biunique eigenfunction pieces in each of the spatial dimensions is finite (i.e., is not infinitesimal or infinite) and all functions are continuous. So, if any spatial dimension is infinite, *M* is infinite and the field is not definable. If all spatial dimensions are finite, we have shown that the field of equation (2) is quantized, hence definable in T. The field is thus definable in T if and only if all the spatial dimensions are finite. We have obtained compactification effectively. Note that this is achieved without invoking boundary conditions. Thus compactification of the spatial dimensions is equivalent to quantization.

8. Borchers, H.-J. and Sen, R. N., Theory of Ordered Spaces II, Commun. Math. Phys., 1999, 204**.**

9. Dyson, F. J., Divergence of Perturbation Theory in Quantum Electrodynamics, Phys.Rev., 1952, 85.

10. Wigner, E. P., The Unreasonable Effectiveness of Mathematics in the Natural Sciences, Comm. Pure and Appl. Math. 1960, 13.

Appendix

ZF - AR + ABR + Constructibility

Extensionality- Two sets with just the same members are equal.

$$\forall x \forall y [\forall z [z \in x \leftrightarrow z \in y] \rightarrow x = y]$$

Pairs- For every two sets, there is a set that contains just them.

$$\forall x \forall y \exists z [\forall w w \in z \leftrightarrow w = x \vee w = y]$$

Union- For every set of sets, there is a set with just all their members.

$$\forall x \exists y \forall z [z \in y \leftrightarrow \exists u [z \in u \wedge u \in x]]$$

Infinity- There is at least one set with members determined in infinite succession

$$\exists \omega^* [0 \in \omega^* \wedge \forall x [x \in \omega^* \rightarrow x \cup \{x\} \in \omega^*]]$$

Power Set- For every set, there is a set containing just all its subsets.

$$\forall x \exists P(x) \forall z [z \in P(x) \leftrightarrow z \subseteq x]$$

Regularity- Every non-empty set has a minimal member (i.e. "weak" regularity).

$$\forall x [\exists y y \in x \rightarrow \exists y [y \in x \wedge \forall z \neg [z \in x \wedge z \in y]]]$$

Replacement- Replacing members of a set one-for-one creates a set (i.e., "bijective" replacement).

Let $\phi(x,y)$ a formula in which x and y are free,

$$\forall z \forall x \in z \exists y [\phi(x,y) \wedge \forall u \in z \forall v [\phi(u,v) \rightarrow u = x \leftrightarrow y = v]] \rightarrow \exists r \forall t [t \in r \leftrightarrow \exists s \in z \phi(s,t)]$$

Constructibility- All the subsets of any ω^* are constructible.

$$\forall \omega^* \exists S[(\omega^*,0) \in S \wedge \forall y E!z[y \neq 0 \wedge y \subseteq \omega^* \wedge (y,z) \in S \leftrightarrow (y \cup m_y - \{m_y\}, z \cup \{z\}) \in S]]$$

where m_y is the minimal member of y.